Wednesday, November 4, 2015

p. 549: 17, 18, 19, 21, 22, 31, 33

Problem 17

Problem. Use partial fractions to find the indefinite integral $\int \frac{x^2-1}{x^3+x}\,dx$.

Solution. The denominator factors as

$$x^3 + x = x(x^2 + 1).$$

Set up the identity

$$\frac{x^2-1}{x^3+x} = \frac{A}{x} + Bx + Cx^2 + 1$$

and solve for A, B, and C.

$$\begin{aligned}\frac{x^2-1}{x^3+x} &= \frac{A}{x} + Bx + Cx^2 + 1,\\ x^2 - 1 &= A(x^2+1) + (Bx + C)(x).\end{aligned}$$

Let $x = 0$:

$$-1 = A.$$

Let $x = 1$:

$$\begin{aligned}0 &= A(2) + (B + C)(1)\\ 0 &= -2 + B + C\\ 2 &= B + C.\end{aligned}$$

Let $x = -1$:

$$\begin{aligned}0 &= A(2) + (-B + C)(-1)\\ 0 &= -2 + B - C,\\ 2 &= B - C.\end{aligned}$$

Solving the last two equations simultaneously, we get $B = 2$ and $C = 0$. Now we can integrate.

$$\begin{aligned}\int \frac{x^2-1}{x^3+x}\,dx &= -\int \frac{1}{x}\,dx + 2\int \frac{x}{x^2+1}\,dx\\ &= -ln|x| + \ln|x^2+1| + C.\end{aligned}$$

Problem 18

Problem. Use partial fractions to find the indefinite integral $\int \frac{6x}{x^3 - 8}\,dx$.

Solution. The denominator factors as

$$x^3 - 8 = (x - 2)(x^2 + 2x + 4).$$

Set up the identity

$$\frac{6x}{x^3 - 8} = \frac{A}{x - 2} + \frac{Bx + C}{x^2 + 2x + 4}$$

and solve for A, B, and C.

$$\begin{aligned} \frac{6x}{x^3 - 8} &= \frac{A}{x - 2} + \frac{Bx + C}{x^2 + 2x + 4}, \\ 6x &= A(x^2 + 2x + 4) + (Bx + C)(x - 2). \end{aligned}$$

Let $x = 2$:

$$\begin{aligned} 12 &= A(12), \\ A &= 1. \end{aligned}$$

Let $x = 0$:

$$\begin{aligned} 0 &= A(4) + (C)(-2) \\ 0 &= 4 - 2C, \\ C &= 2. \end{aligned}$$

Let $x = 1$:

$$\begin{aligned} 6 &= A(7) + (B + C)(-1) \\ 6 &= 7 - B - 2, \\ B &= -1. \end{aligned}$$

Now we can integrate.

$$\begin{aligned} \int \frac{6x}{x^3 - 8}\,dx &= \int \frac{1}{x - 2}\,dx - \int \frac{x - 2}{x^2 + 2x + 4}\,dx \\ &= \ln|x - 2| - \int \frac{x - 2}{(x + 1)^2 + 3}\,dx. \end{aligned}$$

In the second integral, let $u = x + 1$, $du = dx$. Then $x = u - 1$.

$$\begin{aligned}
\int \frac{x-2}{(x+1)^2+3}\, dx &= \int \frac{(u-1)-2}{u^2+3}\, du \\
&= \int \frac{u-3}{u^2+3}\, du \\
&= \frac{1}{2}\int \frac{2u}{u^2+3}\, dx - 3\int \frac{1}{u^2+3}\, du \\
&= \frac{1}{2}\ln|u^2+3| - 3\int \frac{1}{u^2+3}\, du \\
&= \frac{1}{2}\ln|x^2+2x+4| - 3\int \frac{1}{u^2+3}\, du
\end{aligned}$$

For the last integral, we could look it up, or we could make the substitution $u = \sqrt{3}v$, $du = \sqrt{3}\, dv$. Then

$$\begin{aligned}
3\int \frac{1}{u^2+3}\, du &= 3\sqrt{3}\int \frac{1}{3v^2+3}\, dv \\
&= \sqrt{3}\int \frac{1}{v^2+1}\, dv \\
&= \sqrt{3}\arctan v + C \\
&= \sqrt{3}\arctan\left(\frac{u}{\sqrt{3}}\right) + C \\
&= \sqrt{3}\arctan\left(\frac{x+1}{\sqrt{3}}\right) + C
\end{aligned}$$

Putting it all together, we get

$$\int \frac{6x}{x^3-8}\, dx = \ln|x-2| - \frac{1}{2}\ln|x^2+2x+4| + \sqrt{3}\arctan\left(\frac{x+1}{\sqrt{3}}\right) + C.$$

Problem 19

Problem. Use partial fractions to find the indefinite integral $\displaystyle\int \frac{x^2}{x^4-2x^2-8}\, dx$.

Solution. The denominator factors as

$$x^4 - 2x^2 - 8 = (x^2-4)(x^2+2) = (x-2)(x+2)(x^2+2).$$

Set up the identity

$$\frac{x^2}{x^4-2x^2-8} = \frac{A}{x-2} + \frac{B}{x+2} + \frac{Cx+D}{x^2+2}$$

and solve for A, B, C, and D.

$$\begin{aligned}\frac{x^2}{x^4-2x^2-8} &= \frac{A}{x-2} + \frac{B}{x+2} + \frac{Cx+D}{x^2+2},\\ x^2 &= A(x+2)(x^2+2) + B(x-2)(x^2+2) + (Cx+D)(x-2)(x+2).\end{aligned}$$

Let $x = 2$:

$$\begin{aligned}4 &= A(4)(6),\\ A &= \frac{1}{6}.\end{aligned}$$

Let $x = -2$:

$$\begin{aligned}4 &= B(-4)(6)\\ B &= -\frac{1}{6}.\end{aligned}$$

Let $x = 0$:

$$\begin{aligned}0 &= A(2)(2) + B(-2)(2) + (D)(-2)(2)\\ 0 &= \frac{2}{3} + \frac{2}{3} - 4D,\\ D &= \frac{1}{3}.\end{aligned}$$

Let $x = 1$:

$$\begin{aligned}1 &= A(3)(3) + B(-1)(3) + (C+D)(-1)(3)\\ 1 &= 9A - 3B - 3C - 3D\\ 1 &= \frac{3}{2} + \frac{1}{2} - 3C - 1\\ 1 &= -3C + 1\\ C &= 0.\end{aligned}$$

Now we can integrate.

$$\begin{aligned}\int \frac{x^2}{x^4-2x^2-8}\,dx &= \frac{1}{6}\int \frac{1}{x-2}\,dx - \frac{1}{6}\int \frac{1}{x+2}\,dx + \frac{1}{3}\int \frac{1}{x^2+2}\,dx\\ &= \frac{1}{6}\ln|x-2| - \frac{1}{6}\ln|x+2| = \frac{1}{3\sqrt{2}}\arctan\frac{x}{\sqrt{2}} + C.\end{aligned}$$

Problem 21

Problem. Use partial fractions to find the indefinite integral $\int \frac{x^2+5}{x^3-x^2+x+3}\,dx$.

Solution. The denominator factors as

$$x^3 - x^2 + x + 3 = (x+1)(x^2 - 2x + 3).$$

Set up the identity

$$\frac{x^2+5}{x^3-x^2+x+3} = \frac{A}{x+1} + \frac{Bx+C}{x^2-2x+3}$$

and solve for A, B, and C.

$$\begin{aligned}\frac{x^2+5}{x^3-x^2+x+3} &= \frac{A}{x+1} + \frac{Bx+C}{x^2-2x+3},\\ x^2+5 &= A(x^2-2x+3) + (Bx+C)(x+1).\end{aligned}$$

Let $x = -1$:

$$\begin{aligned}6 &= A(6),\\ A &= 1.\end{aligned}$$

Let $x = 0$:

$$\begin{aligned}5 &= A(3) + C(1)\\ 5 &= 3 + C,\\ C &= 2.\end{aligned}$$

Let $x = 1$:

$$\begin{aligned}6 &= A(2) + (B+C)(2)\\ 6 &= 2 + 2B + 4,\\ B &= 0.\end{aligned}$$

Now we can integrate.

$$\begin{aligned}\int \frac{x^2+5}{x^3-x^2+x+3}\,dx &= \int \frac{1}{x+1}\,dx + 2\int \frac{1}{x^2-2x+3}\,dx\\ &= \ln|x+1| + 2\int \frac{1}{(x-1)^2+2}\,dx.\end{aligned}$$

In the remaining integral, let $u = x - 1$, $du = dx$. We get

$$\begin{aligned} 2\int \frac{1}{(x-1)^2+2}\, dx &= 2\int \frac{1}{u^2+2}\, du \\ &= 2 \cdot \frac{1}{\sqrt{2}} \arctan \frac{u}{\sqrt{2}} + C \\ &= \sqrt{2} \arctan\left(\frac{x-1}{\sqrt{2}}\right) + C. \end{aligned}$$

Combining the two integrals, we get

$$\int \frac{x^2+5}{x^3-x^2+x+3}\, dx = \ln|x+1| + \sqrt{2}\arctan\left(\frac{x-1}{\sqrt{2}}\right) + C.$$

Problem 22

Problem. Use partial fractions to find the indefinite integral $\int \frac{x^2+6x+4}{x^4+8x^2+16}\, dx$.

Solution. The denominator factors as $x^4 + 8x^2 + 16 = (x^2+4)^2$. Set up the identity

$$\frac{x^2+6x+4}{x^4+8x^2+16} = \frac{Ax+B}{x^2+4} + \frac{Cx+D}{(x^2+4)^2}$$

and solve for A, B, C, and D. We get

$$x^2 + 6x + 4 = (Ax+B)(x^2+4) + (Cx+D).$$

Let $x = 0$:

$$\begin{aligned} 4 &= B(4) + D \\ &= 4B + D. \end{aligned}$$

Let $x = 1$:

$$\begin{aligned} 11 &= (A+B)(5) + (C+D) \\ &= 5A + 5B + C + D. \end{aligned}$$

Let $x = -1$:

$$\begin{aligned} -1 &= (-A+B)(5) + (-C+D) \\ &= -5A + 5B - C + D. \end{aligned}$$

Let $x = 2:$

$$\begin{aligned} 20 &= (2A + B)(8) + (2C + D) \\ &= 16A + 8B + 2C + D. \end{aligned}$$

Solving the system, we get $A = 0$, $B = 1$, $C = 6$, and $D = 0$. Now we can integrate.

$$\begin{aligned} \int \frac{x^2 + 6x + 4}{x^4 + 8x^2 + 16}\, dx &= \int \frac{1}{x^2 + 4}\, dx + \int \frac{6x}{(x^2 + 4)^2}\, dx \\ &= \frac{1}{2} \arctan \frac{x}{2} + \int \frac{6x}{(x^2 + 4)^2}\, dx. \end{aligned}$$

In the remaining integral, let $u = x^2 + 4$, $du = 2x\, dx$. We get

$$\begin{aligned} 3 \int \frac{2x}{(x^2 + 4)^2}\, dx &= 3 \int \frac{1}{u^2}\, du \\ &= -\frac{3}{u} + C \\ &= -\frac{3}{x^2 + 4} + C. \end{aligned}$$

The solution is

$$\int \frac{x^2 + 6x + 4}{x^4 + 8x^2 + 16}\, dx = \frac{1}{2} \arctan \frac{x}{2} - \frac{3}{x^2 + 4} + C.$$

Problem 31

Problem. Use substitution and partial fractions to find the indefinite integral $\displaystyle\int \frac{e^x}{(e^x - 1)(e^x + 4)}\, dx$.

Solution. Let $u = e^x$, $du = e^x\, dx$. The problem becomes

$$\int \frac{e^x}{(e^x - 1)(e^x + 4)}\, dx = \int \frac{1}{(u - 1)(u + 4)}\, du.$$

Set up the identity

$$\frac{1}{(u - 1)(u + 4)} = \frac{A}{u - 1} + \frac{B}{u + 4}.$$

$$\begin{aligned} \frac{1}{(u - 1)(u + 4)} &= \frac{A}{u - 1} + \frac{B}{u + 4} \\ 1 &= A(u + 4) + B(u - 1). \end{aligned}$$

Let $u = 1:$

$$\begin{aligned} 1 &= A(5), \\ A &= \frac{1}{5}. \end{aligned}$$

Let $u = -4:$

$$1 = B(-5), \tag{1}$$

$$B = -\frac{1}{5}. \tag{2}$$

Now we can integrate.

$$\begin{aligned} \int \frac{1}{(u-1)(u+4)}\,du &= \frac{1}{5}\int \frac{1}{u-1}\,du - \frac{1}{5}\int \frac{1}{u+4}\,du \\ &= \frac{1}{5}\ln|u-1| - \frac{1}{5}\ln|u+4| + C \\ &= \frac{1}{5}\ln|e^x-1| - \frac{1}{5}\ln|e^x+4| + C. \end{aligned}$$

Problem 33

Problem. Use substitution and partial fractions to find the indefinite integral $\displaystyle\int \frac{e^x}{(x^{2x}+1)(e^x-1)}\,dx.$

Solution. Let $u = e^x$, $du = e^x\,dx$. Then

$$\int \frac{e^x}{(x^{2x}+1)(e^x-1)}\,dx = \int \frac{1}{(u^2+1)(u-1)}\,du.$$

Set up the identity

$$\frac{1}{(u-1)(u+4)} = \frac{A}{u-1} + \frac{B}{u+4}.$$

$$\begin{aligned} \frac{1}{(u^2+1)(u-1)} &= \frac{Au+B}{u^2+1} + \frac{C}{u-1} \\ 1 &= (Au+B)(u-1) + C(u^2+1). \end{aligned}$$

Let $u = 1:$

$$\begin{aligned} 1 &= C(2), \\ C &= \frac{1}{2}. \end{aligned}$$

Let $u = 0:$

$$\begin{aligned} 1 &= B((-1) + C(1) && (3)\\ &= -B + C, && (4)\\ B &= -\frac{1}{2}. && (5) \end{aligned}$$

Let $u = -1:$

$$\begin{aligned} 1 &= (-A + B)(-2) + C(2)\\ &= 2A - 2B + 2C\\ &= 2A + 1 + 1,\\ A &= -\frac{1}{2}. \end{aligned}$$

Now we can integrate.

$$\begin{aligned} \int \frac{1}{(u^2+1)(u-1)}\,du &= -\frac{1}{2}\int \frac{u+1}{u^2+1}\,du + \frac{1}{2}\int \frac{1}{u-1}\,du\\ &= -\frac{1}{4}\int \frac{2u}{u^2+1}\,du - \frac{1}{2}\int \frac{1}{u^2+1}\,du + \frac{1}{2}\int \frac{1}{u-1}\,du\\ &= -\frac{1}{4}\ln|u^2+1| - \frac{1}{2}\arctan u + \frac{1}{2}\ln|u-1| + C\\ &= -\frac{1}{4}\ln|e^{2x}+1| - \frac{1}{2}\arctan e^x + \frac{1}{2}\ln|e^x-1| + C. \end{aligned}$$